P395.

4. Sol:

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}] = W(t_0)\, e^{\int_{t_0}^{t} -p(s)ds}$$

$$W[y_1, y_2] = W(t_1)\, e^{-\int_{t_1}^{t} p(s)ds}$$

if we take the same initial value, then $W[y_1, y_2] = W[\vec{x}^{(1)}, \vec{x}^{(2)}]$,

otherwise, $W[y_1, y_2] = C\, W[\vec{x}^{(1)}, \vec{x}^{(2)}]$ for $C \neq 0$.

P406.

29. Sol:

Set $x_1 := y$, $x_2 := y'$,

then $x_1' = y'$, $x_2' = y''$.

$$\vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = A \begin{pmatrix} y \\ y' \end{pmatrix}$$

Since $ay'' + by' + cy = 0$.

$$\Rightarrow y'' = -\frac{by' + cy}{a}$$

$$\Rightarrow \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.$$

$$\det(\lambda I - A) = \begin{pmatrix} \lambda & -1 \\ \frac{c}{a} & \lambda + \frac{b}{a} \end{pmatrix} = 0 \Rightarrow \lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} = 0$$

i.e. $a\lambda^2 + b\lambda + c = 0$.

31. Sol:

a> when $\alpha = \frac{1}{2}$, $A = \begin{pmatrix} -1 & -1 \\ -\frac{1}{2} & -1 \end{pmatrix}$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda+1 & 1 \\ \frac{1}{2} & \lambda+1 \end{pmatrix} = (\lambda+1)^2 - \frac{1}{2} = 0$$

$$\Rightarrow \lambda = -1 \pm \sqrt{\frac{1}{2}} = \frac{-2 \pm \sqrt{2}}{2}$$

$$(\lambda_1 I - A)x = 0 \Leftrightarrow \begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ \frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \text{set } x_1 = 1 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$(\lambda_2 I - A)x = 0 \Leftrightarrow \begin{pmatrix} -\frac{\sqrt{2}}{2} & 1 \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

i.e. $\lambda_1 = \frac{-2+\sqrt{2}}{2}$ $\quad r_1 = \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$

$\lambda_2 = \frac{-2-\sqrt{2}}{2}$ $\quad r_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ $\qquad \lambda_1\lambda_2 > 0 \Rightarrow$ node.

$$\vec{x} = C_1 \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} e^{\frac{-2+\sqrt{2}}{2}t} + C_2 \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} e^{-\frac{2+\sqrt{2}}{2}t}$$

b> when $\alpha = 2$, $\quad \lambda_1 = -1+\sqrt{2}$, $\quad r_1 = \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}$

$\lambda_2 = -1-\sqrt{2}$ $\quad r_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ $\qquad \lambda_1\lambda_2 < 0 \Rightarrow$ saddle pt.

$$\vec{x} = C_1 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2})t} + C_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-(1+\sqrt{2})t}.$$

c> $\lambda_1 = -1+\sqrt{\alpha}$ $\quad \lambda_2 = -1-\sqrt{\alpha}$

$\lambda_1 \lambda_2 = 1-\alpha$, hence $\alpha = 1$ will change type.

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19. Sol:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda-5 & 3 & 2 \\ -8 & \lambda+5 & 4 \\ 4 & -3 & \lambda-3 \end{pmatrix}$$

$$= (\lambda^2-25)(\lambda-3) + 48 + 48 - 8(\lambda+5) + 12(\lambda-5) + 24(\lambda-3)$$

$$= \lambda^3 - 3\lambda^2 - 25\lambda + 75 + 96 + 28\lambda - 172$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0.$$

i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$(\lambda I - A)x = 0 \iff \begin{pmatrix} -4 & 3 & 2 \\ -8 & 6 & 4 \\ 4 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} -4 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

take $x_1 = 1$, $x_2 = 0 \Rightarrow x_3 = 2$

$x_1 = 0$ $\quad x_2 = 2 \Rightarrow x_3 = -3$.

i.e. $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\quad \vec{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$ are two eigenvectors.

$$\Rightarrow \vec{x}^{(1)}(t) = c_1 e^{\lambda_1 t} \vec{\xi}^{(1)} = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\vec{x}^{(2)}(t) = c_2 e^{\lambda_2 t} \vec{\xi}^{(2)} = c_2 e^t \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

b> Assume that $\vec{x} = \vec{\xi}\, te^t + \vec{\eta}\, e^t$.

then $\vec{x}\,' = e^t\left((1+t)\vec{\xi} + \vec{\eta}\right) = te^t \vec{\xi} + e^t(\vec{\xi} + \vec{\eta})$.

$$= A\vec{x} = A\left[\vec{\xi}\, te^t + \vec{\eta}\, e^t\right]$$

Since it holds for all t, we have:

$$A\vec{\xi} = \vec{\xi} \iff (A-I)\vec{\xi} = 0$$

$$A\vec{\eta} = \vec{\xi} + \vec{\eta} \iff (A-I)\vec{\eta} = \vec{\xi}.$$

c> Notice that $(A-I)^2\vec{\eta} = (A-I)\vec{\xi} = 0$:

and $(A-I)^2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$.

hence we may choose $\vec{\eta}$ arbitrarily.

d> take $\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then $\vec{\xi} = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}$

$$(A-I)\vec{\xi} = (A-I)^2\vec{\eta} = 0.$$

~~e>~~ $\cancel{\vec{\Psi}(t) = c_1 e^t \vec{\xi}^{(1)} + c_2 e^t \vec{\xi}^{(2)} + c_3\left(e^t\vec{\eta} + te^t\vec{\xi}\right)}$.

e> $\Psi(t) = e^t \begin{pmatrix} 1 & 0 & -2t \\ 0 & 2 & -4t \\ 2 & -3 & 2t+1 \end{pmatrix}$

f> : $T = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & 0 \\ 2 & 2 & 1 \end{pmatrix}$ $\quad T^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & 0 \\ -2 & \frac{3}{2} & 1 \end{pmatrix}$ $\quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.